DIFFRACTION OF RADIO WAVES
AROUND THE EARTH'S SURFACE

V. Fock

Translated by Lt. Cdr. P.N. Hiles

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CONTENTS

INTRODUCTION

A wide selection of literature is devoted to the problem of diffraction of radio waves around the surface of the Earth; a review of the most recent studies can be found in Academician B.A. Vvedenskij's article[2]. Interest in this question is justified in that, at short distances of the order of a few hundred kilometres, refraction of radio waves in the ionized layers of the atmosphere can be ignored, while diffraction plays the decisive role in radio-wave propagation.

Despite the fact that a firm solution of the question of diffraction from the globe has already been known for some decades, a practically applicable approximate solution has not, up to now, been obtained. In the present work we intend to fill this gap.

1. STATEMENT OF THE PROBLEM AND ITS SOLUTION IN SERIES FORM

By r, ϑ and φ we mean spherical coordinates with their origin at the centre of the Earth. If we ignore the irregularities of the Earth's surface, then its equation will take the form $r = a$, where $a =$ Earth's radius. Let us place a vertical electric dipole ($b > a$) at the point $r = b$, $\vartheta = 0$. Rejecting in the field components the time dependent factor $e^{-i\omega t}$, we can express it by the Hertz function U, which will depend only $\cdot$ r and on ϑ. Having by k denoted the absolute value of the wave vector, $\nu \cdot$ shall have, for the field in air, the expression

$$
\left.
\begin{aligned}
E_r &= \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \frac{\partial U}{\partial \vartheta} \right), \\[2mm]
E_\vartheta &= - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial} \right); \\[2mm]
H_\varphi &= - ik \frac{\partial U}{\partial \vartheta} ;
\end{aligned}
\right\}
\tag{1,01}
$$

while the remaining components of the field will equal zero. Similar expressions will apply for the field within the earth. The U function must, where $r > a$, satisfy the equation

$$
\Delta U + k^2 U = 0 \tag{1,02}
$$

and the condition of radiation to infinity

$$
\lim_{r \to \infty} \left(\frac{d r U}{\partial r} - ik r U \right) = 0 \tag{1,03}
$$

If $b > a$, in such a way that the source (dipole) is above the Earth's surface and not on the surface itself, then at Point $r = b$, $\vartheta = 0$ the U function must have the property:

$$U = \frac{e^{ikR}}{R} + U^*, \qquad (1,04)$$

where

$$R = \sqrt{r^2 + b^2 - 2rb \cos \vartheta} \qquad (1,05)$$

is the distance from the source, while the U^* function remains finite when $kR \to 0$. On the Earth's surface the Hertz function U must satisfy the limiting conditions guaranteeing continuity of the components E_ϑ and H_φ at the interface.

If we denote the Hertz function inside the Earth as U_2 then these limiting conditions will take the form

$$k^2 U = k_2^2 U_2; \qquad \frac{\partial}{\partial r}(rU) = \frac{\partial}{\partial r}(rU_2) \quad \text{where } r = a . \qquad (1,06)$$

The U_2 function must, when $0 \le r \le a$ (inside the Earth), satisfy an equation similar to (1,02) and remain finite.

The value k_2, introduced in (1,06) and later formulae, is determined from the equation

$$k_2^2 = \varepsilon k^2 + i\frac{4\pi\sigma}{c} \qquad (1,07)$$

and the condition $\text{Im}(k_2) > 0$. In place of the conductance of the Earth σ it is convenient to bring in length l, characterizing the Earth's specific resistance, having assumed

$$l = \frac{c}{4\pi\sigma} . \qquad (1,08)$$

For sea water the value of l varies from 0.05 cm (very salt water) to 0.5 cm (slightly salt water); for the earth this length is hundreds

and thousands of times greater. If we bring in the complex dialectric constant of the Earth

$$\eta \ = \ \epsilon \ + \ i \ \frac{\lambda}{2\pi l} \ , \qquad (1,09)$$

then

$$k_2 \ = \ k \ \sqrt{\eta} \ . \qquad (1,10)$$

A solution of the problem posed in the form of series is well known; Here we inscribe the necessary formulae without dwelling on their derivation.

We assume

$$\psi_n(x) \ = \ \sqrt{\frac{\pi x}{2}} J_{n+\frac{1}{2}}(x); \quad \zeta_n(x) \ = \ \sqrt{\frac{\pi x}{2}} H^{(1)}_{n+\frac{1}{2}}(x), \quad (1,11)$$

where $J^\nu(x)$ is the Bessel function, while $H^{(1)}_\nu(x)$ is a first order Hankel function. These functions are connected by the relationship

$$\psi_n(x)\zeta'_n(x) \ - \ \psi_n(x)\zeta_n(x) \ = \ i. \qquad (1,12)$$

Additionally, we introduce the particular specification for the logarithmic derivative from $\psi_n(x)$;

$$\chi_n(x) \ = \ \frac{\psi'_n(x)}{\psi_n(x)} \ . \qquad (1,13)$$

As is seen from (1,01), the field at the Earth's surface is expressed by the values:

$$U_a \ = \ (U)_{r=a}; \qquad U_a{'} \ = \ \left(\frac{\partial}{\partial r}(rU)\right)_{r=a}. \qquad (1,14)$$

For these values we obtain the series:

$$U_a = - \frac{1}{kab} \sum_{n=o} \frac{(2n + 1)\zeta_n(kb)}{\zeta_n'(ka) - \frac{k}{k_2}\chi_n(k_2a)\zeta_n(ka)} P_n(\cos\theta); \qquad (1,15)$$

$$U_a' = - \frac{k}{k_2 b} \sum_{n=o} \frac{(2n+1)\zeta_n(kb)\chi_n(k_2a)}{\zeta_n'(ka) - \frac{k}{k_2}\chi_n(k_2a)\zeta_n(ka)} P_n(\cos\theta), \qquad (1,16)$$

ordered according to LeGendre's polynomials. Our task consists of an approximate summation of these series.

2. SUMMATION FORMULA

The sums which we have to calculate have the form

$$S = \sum_{\nu=\frac{1}{2}, \frac{3}{2}\cdots} \nu\varphi(\nu) P_{\nu-\frac{1}{2}}(\cos\theta). \qquad (2,01)$$

In the Equ.(1,14), the function $\varphi(\nu)$ with an accuracy approaching the factor equal to

$$\varphi(\nu) = \frac{\zeta_{\nu-\frac{1}{2}}(kb)}{\zeta_{\nu-\frac{1}{2}}'(ka) - \frac{k}{k_2} \chi_{\nu-\frac{1}{2}}(k_2a)\zeta_{\nu-\frac{1}{2}}(ka)} \qquad (2,02)$$

In Equ. (1,15) it differs from (2,02) by the factor $\chi_{\nu-\frac{1}{2}}(k_3a)$.

For a direct calculation of the sum it would be necesssary to take the number of terms approximately equal to $2ka$, i.e. equal to double the number of waves present on the Earth's surface. Since this number is enormous, it is evident that an immediate summation is impossible. To compute the sum S it is necessary, having taken advantage of the fact that $\varphi(\nu)$ is an analytical function, to convert this sum to an integral which can be computed by one or other of the approximation methods. A similar conversion was first proposed by Watson in 1918 and adopted by many authors. However, all the authors attempted to apply the converted expression to the sum of deductions, whereas our intention is to separate out the main terms most easily subject to investigation and to estimate the remainder. In this

method of omputation the main terms are not predetermined.

For the completion of our conversion it was essential to keep in mind the following general properties of the function $\varphi(v)$. It is an analytic function from v, meromorphic in the right half-plane; it has poles only in the first quarter, while in the fourth quarter it is holomorphic. At infinity $\varphi(v)$ declines so rapidly that all the integrals being investigated converge.

As seen in Equation (2,01) the LeGendre's functions can be expressed by the function

$$G_v = \frac{\Gamma\left(v + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(v + 1)} \, F\left(\frac{1}{2}, \frac{1}{2}, \, v + 1, \frac{ie^{-i\vartheta}}{2\sin\vartheta}\right), \quad (2,03)$$

where F is the sign for the hypergeometric function. By G_v^* and $P_{v-\frac{1}{2}}^*$ denote the result of a change in G_v and in $P_{v-\frac{1}{2}} = P_{v-\frac{1}{2}}(\cos\vartheta)$ of the value ϑ in $\pi - \vartheta$. We shall then have

$$P_{v-\frac{1}{2}} = \frac{1}{\pi\sqrt{2\sin\vartheta}} \left[e^{iv\vartheta - i\frac{\pi}{4}} G_v^* + e^{-iv\vartheta + i\frac{\pi}{4}} G_v \right] \quad (2,04)$$

As can be seen from (2,03), if v lies within a certain sector including the negative axis, and if $|v\sin\vartheta|$ is large, the function G_v (and also G_v^*) approximately equal

$$G_v \sim \sqrt{\frac{\pi}{v}} \quad (2,05)$$

Substituting (2,05) in (2,04), we obtain the known asymptotic expression for $P_{v-\frac{1}{2}}$. If $B(v)$ denoted the first term in Equation (2,04)

$$B(v) = \frac{1}{\pi\sqrt{2\sin\vartheta}} e^{iv\vartheta - i\frac{\pi}{4}} G_v^*, \quad (2,06)$$

one can substantiate the equation

$$P^*_{v-\frac{1}{2}} = e^{i(v-\frac{1}{2})\pi} P_{v-\frac{1}{2}} + 2i \cos v\pi B(v) \qquad (2,07)$$

which we shall make use of later. We note that the function $B(v)$ is holomorphic in the right half-plane.

We examine in the plane of the complex variable three contours. Firstly, the loop C_1 around the origin of the coordinates alleviating the positive material axis and essential in the positive direction (anti-clockwise). Secondly, the broken line C_2 encompassing the first quadrant and running from left to right (in its horizontal part passing a little higher than the material axis). Thirdly, the straight line C_3, passing through the origin of the coordinates, lying in the second and fourth quadrants and inclined at a small angle to the imaginary axis; this straight line runs from top to bottom.

We can write the sum S in the form

$$S = \frac{1}{2} \int_{C_1} v\varphi(v) \sec v\pi P^*_{v-\frac{1}{2}} dv, \qquad (2,09)$$

since the integral on the right comes to deductions at the points $v = n + \frac{1}{2}$. Since the function $\varphi(v)$ is holomorphic in the fourth quadrant, we can substitute Contour C_1 for Contours C_2 and C_3 and write

$$S = -\frac{i}{2} \int_{C_2} v\varphi(v) \sec v\pi P^*_{v-\frac{1}{2}} dv + \frac{i}{2} \int_{C_3} v\varphi(v) \sec v\pi P^*_{v-\frac{1}{2}} dv. \qquad (2,09)$$

The normal conversion of the sum comes down to this: the integral at C_3 is assumed to equal zero because of the smallness of the uneven part of $\varphi(v)$ (its definition will be given below), and the integral at C_2 boils down to the sum of deductions. but we shall take a step further and break up C_2 into main term and correction [term]. Substituting in this integral the expression (2,07) for $P^*_{v-\frac{1}{2}}$, we shall have

$$S = S_1 + S_2 + S_3, \qquad (2,10)$$

$$S_1 = \int_C \nu\varphi(\nu)B(\nu)\,d\nu, \qquad (2,11)$$

$$S_2 = -\frac{1}{2}\int_{C_2} \nu\varphi(\nu)\,\sec\nu\pi e^{i\nu\pi}\,P_{\nu-\frac{1}{2}}\,d\nu, \qquad (2,12)$$

$$S_3 = \frac{i}{2}\int_{C_3} \nu\varphi(\nu)\,\sec\nu\pi P^{*}_{\nu-\frac{1}{2}}\,d\nu. \qquad (2,13)$$

In integral S_1 the subintegral function is already without poles on the material axis (and also in the fourth quarter), therefore Contours C_2 and C_2 [sic] for it are equivalent. We denote by C any contour equivalent to C_2 and C_3.

The presentation of the value S in the form of three integrals (2,10) is accurate; in the derivation nothing was neglected. But the definition of integral S_2 and S_3 indicates that these integrals are minute in comparison with S_1. In truth, if we are to compute the integral as a sum of deductions in the poles $\varphi(\nu)$ then we are convinced that, in relation to S_1, it will be of the order of

$$\left| e^{2i\nu_1(\pi-\vartheta)} \right|. \qquad (2,14)$$

Where ν_1 is closest to the material axis of the pole $\varphi(\nu)$. The imaginary part of ν_1 is positive and at high values of ka will be

$$Im(\nu_1) \approx c(k\alpha)^{1/3}, \qquad (2,15)$$

where c is a number of the order of unity (for the absolute conductor $c = 0.70$). Since $k\alpha$ is a very large number, of the order of millions

(for $\lambda = 40$ m, $ka = 1,000.000$), it is apparent that the value (2,15) will be large (for instance equal to 70), while the value (2,14) will be very small.(In our problem ϑ may not be close to π, since, then, in consequence of the necessity to consider the effects of the ionized layers of the atmosphere, our equations would generally cease to be applicable). As far as the integral S_3 is concerned, its value is determined by the uneven part of the function $\varphi(v)$. But the uneven part of $\varphi(v)$ will be of the order of

$$\left| e^{2ik_2 a} \right|. \tag{2,16}$$

And since the imaginary part $k_2 a$ is positive and extremely large, then the value of (2,16) will be an inexpressibly small number.

The following physical picture gives a graphic presentation of the smallness of integrals S_2 and S_3. Integral S_2 is the amplitude of a wave travelling without a break (as a result of a single diffraction) around the world once or several times. Integral S_3 is the amplitude of a wave penetrating right through the Earth with the attenuation normal to the Earth. Obviously, both these integrals are minute in comparison with the amplitude of a wave travelling from the source through the air by the shortest path.

Thus, with all accuracy attainable by the statement of the physical problem, the sum S, determined by the equation (2,01), may be expressed in the form of an integral S_1, which, after substituting for the equation (2,06), takes the form

$$S_1 = \frac{e^{-i\frac{\pi}{4}}}{\pi \sqrt{2 \sin \vartheta}} \int_C v\varphi(v)e^{iv\vartheta}G^*_v \, dv. \tag{2,17}$$

3. CALCULATION OF THE HERTZ FUNCTION FOR THE ILLUMINATED REGIONS

If we understand by $\varphi(v)$ the equation (2,02), then the connection between sum S and the value U_a will be

$$U_a = -\frac{2}{kab} S. \tag{3,01}$$

Therefore our approximate expression for U_a is written

$$U_a = \frac{2e^{i\frac{3\pi}{4}}}{\pi kab \sqrt{2 \sin \vartheta}} \int_C \nu \varphi(\nu) e^{i\nu\vartheta} G^*_\nu d\nu \qquad (3,02)$$

The position of the main integration section in Equation (3,02) will depend on which point of the integral is being computed. Generally speaking, the main section will lie close to $\nu = \nu_o$, where

$$\nu_o = kh_c = k \frac{ab \sin \vartheta}{\sqrt{a^2 + b^2 - 2ab \cos \vartheta}} \qquad (3,03)$$

The value h_c is the length of the perpendicular dropped from the centre of the Earth to a beam [of light] (i.e. to a straight line joining the source to the point of observation).

For an approximate computation of Integral U_a it is essential to find for G^*_ν and $\varphi(\nu)$ the asymptotic expressions which will be applicable in the main section. Since the values ν_o and $\nu_o\vartheta$ are large in comparison with unity, we can, in accordance with (2,05), assume

$$G^*_\nu = \sqrt{\frac{\pi}{\nu}}. \qquad (3,04)$$

For the Hankel functions entering into $\varphi(\nu)$ one may attempt to use Debye's expression

$$\zeta_{\nu-\frac{1}{2}}(\rho) = \frac{1}{\sqrt[4]{1 - \frac{\nu^2}{\rho^2}}} e^{i(\xi - \frac{\pi}{4})} \qquad (3,05)$$

where

$$\xi = \int_\rho \sqrt{1 - \frac{\nu^2}{\rho^2}} \, d\rho. \qquad (3,06)$$

... expressions are used in conditions where

$$ | \nu^2 - \nu^2 | > \rho^{2/3} . \tag{3,07} $$

As far as the function $\tau_{\nu-\frac{1}{2}}(k,a)$ is concerned, in the proximity of $\nu = \nu_0$ the expression used for it, with sufficient accuracy, is

$$ \tau_{\nu-\frac{1}{2}}(k,a) = - i \sqrt{1 - \frac{\nu^2}{k_2^2 a^2}} . \tag{3,08} $$

To explain in what conditions the disparity (3,07) is used, we denote by γ the angle between the vertical at the point of observation and the direction to the source and introduce the parameter

$$ \rho = \left(\frac{ka}{2} \right)^{\frac{1}{3}} \cos \gamma. \tag{3,09} $$

It is easy to see that for $\nu = \nu_0$, $\rho = ka$, the disparity (3,07) is equivalent to the condition such that p is a large positive number. Such values of p are applicable to the illuminated region. Values of p of the order of unity (positive and negative) are applicable to the twilight region; where $p = 0$ it gives the edge of the geometric shadow (the horizon line). Negative and large values in relation to absolute p values are applicable to the shaded region.

In this paragraph we examine the condition of large positive values of p (illuminated region); other conditions will be examined in subsequent paras.

When $p \gg 1$, as we have seen, Debye expressions are applicable to the Hankel functions. Substituting them in (3,02) and using (3,04) and (3,08) we obtain the integral

$$ U = \frac{2e^{i\frac{\pi}{4}}}{kab\sqrt{\pi \sin \vartheta}} \sqrt{\frac{b}{a}} \int_0^b \sqrt{\frac{k^2a^2 - \nu^2}{k^2b^2 - \nu^2}} \cdot \frac{e^{i\omega} \sqrt{\nu} \, d\nu}{\sqrt{1 - \frac{\nu^2}{k^2a^2}} + \frac{k}{k_2}\sqrt{1 - \frac{\nu^2}{k_2^2a^2}}} \tag{3,10} $$

where

$$\omega = \int_{ka}^{kb} \sqrt{1 - \frac{\nu^2}{\rho^2}} \, d\rho + \nu \vartheta .$$ (3,11)

In conditions where

$$kh \cos \gamma \gg 1,$$ (3,12)

where $h = b - a$ is the height of the source above the Earth, Equation (3,10) can be computed according to the fixed phase method, obtaining the "reflecting formula"

$$U_a = \frac{e^{i\kappa R}}{R} \, W$$ (3,13)

where

$$R = \sqrt{a^2 + b^2 - 2ab \cos \delta}$$ (3,14)

is the distance from the source, while W is the "attenuation function", which in our case is equal to

$$W = \frac{L}{1 + \frac{k}{k^2} \sqrt{1 - \frac{k^2}{k_2^2} \sin^2 \gamma} \cdot \sec \gamma} .$$ (3,15)

The value of U_a', determined by the series (1,16), differs, in our approximation, only by the constant multiple from U_a, namely

$$U_a' = - \frac{ik \cdot a}{k_2} \sqrt{1 - \frac{k^2}{k_2^2} \sin^2 \gamma} \cdot U_a .$$ (3,16)

The last equation is correct, not only for the illuminated region, but in all cases.

If condition (3,12) is not achieved, then the denominator and subintegral equation (3,10) will not be a slowly changing function. If we assume that the conditions are fulfilled,

$$1 \ll \frac{R^2}{h^2} \ll (ka)^{2/3}, \tag{3,17}$$

$$1 \ll kR \cdot \frac{a}{h} \tag{3,18}$$

(the result of which is the disparity $p \gg 1$), thus the equation (3,10) can be computed approximately by introducing a new variable

$$\mu = \sqrt{1 - \frac{v^2}{k^2 a^2}} \tag{3,19}$$

For the function W in (3,13) the approximate expression

$$W = e^{-i\frac{3\pi}{4}} \sqrt{\frac{2kR}{\pi}} \int_{\Gamma} e^{-i\frac{kR}{2}(\mu-\mu_0)^2} \frac{\mu d\mu}{\mu + \frac{k}{k_2}}, \tag{3,20}$$

is obtained, where

$$\mu_0 = \frac{h}{R}, \tag{3,21}$$

is the inclination of the ray to the horizon and Contour Γ is a straight line passing through the point $\mu = \mu_0$ from the quarter in the second quarter of the plane μ (more precisely $\mu - \mu_0$). The equation (3,20) can be computed without further neglect and gives the known Weyl-van der Pol formula. If we assume

$$\cdot = e^{i\frac{\pi}{4}} \frac{k}{k_2} \sqrt{\frac{kR}{2}}; \qquad = e^{i\frac{\pi}{4}} \frac{h}{R} \sqrt{\frac{kR}{2}}, \tag{3,22}$$

then we shall have

$$W = 2 - 4\pi e^{-(\sigma+\tau)^2} \int_{i\infty}^{\sigma+\sigma i} e^{\alpha'^2} d\alpha \; . \qquad (3.23)$$

In order to obtain, from our expressions U_α and U_α', the formula for the field, it is necessary to differentiate these expressions according to ϑ. But the derivatives according to ϑ are easy to compute, since, in (3,13), it is possible to consider all multiples, apart from $e^{i\kappa R}$ as constants.

4. ASYMPTOTIC EXPRESSIONS FOR THE HANKEL FUNCTION

Later, we shall have to investigate the case where the observation point lies in the twilight region.

This condition is characterized by the values of the p parameter (positive and negative) of the order of unity. Since the disparity (3,07) ceases to be useful for these values of p in the main section of integration, the Debye expressions (3,05) for the Hankel function become inapplicable and must be exchanged for others. New expressions for the Hankel functions, appropriate to our task, may be obtained from asymptotic expressions quoted in our former work[3] and also from formulae in Watson's well known book[4], but it is simpler to derive them direct.

Unknown expressions give a description of Hankel functions via the function $w(t)$, determined by the integral

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{l} e^{tz - \frac{1}{3} z^3} dz, \qquad (4,01)$$

where the contour l runs from infinity to zero along the straight arc $z = 2\pi/3$ and from zero to infinity along the straight arc $z = 0$ (along the positive material axis). The function $w(t)$ satisfies the differential equation

$$w'(t) = tw(t) \qquad (4,02)$$

nd the initial conditions

$$w(0) = \frac{2\sqrt{\pi}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} e^{i\frac{\pi}{6}} = 1.0899290710 + i\,0.6292708425,$$

$$w'(0) = \frac{2\sqrt{\pi}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)} e^{-i\frac{\pi}{6}} = 0.7945704238 - i\,0.4587454481.$$

(4,03)

It is the entire transcendental function which breaks down into a step series of the form

$$w(t) = w(0)\left\{1 + \frac{t^3}{2\cdot3} + \frac{t^6}{(2\cdot5)(3\cdot6)} + \frac{t^9}{(2\cdot5\cdot8)(3\cdot6\cdot9)} + \cdots\right\} +$$

(4,04)

$$+ w'(0)\left\{t + \frac{t^4}{3\cdot4} + \frac{t^7}{(3\cdot6)(4\cdot7)} + \frac{t^{10}}{(3\cdot6\cdot9)(4\cdot7\cdot10)} + \cdots\right\}.$$

If we separate in $w(t)$ the material and imaginary parts (for the material t) and assume

$$w(t) = u(t) + iv(t) \qquad (4,05)$$

then $u(t)$ and $v(t)$ will be two independent integrals of the equation (4.02), connected by the relationship

$$u'(t)v(t) - u(t)v'(t) = 1 \qquad (4,06)$$

Asymptotic expressions for these functions for large negative t values are obtained by separating the material and imaginary parts in the equations

$$w(t) = e^{i\frac{\pi}{4}}(-t)^{-1/4} e^{2/3\,i(-t)^{3/2}} \qquad (4,07)$$

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$$w'(t) = e^{-i\frac{\pi}{4}}(-t)^{1/4} e^{\frac{2}{3}j(-t)^{3/2}}. \qquad (4,08)$$

For large positive t values asymptotic expressions for $u(t)$, $v(t)$ and their derivatives have the form:

$$u(t) = t^{-1/4} e^{\frac{2}{3}t^{3/2}} \quad ; \quad u'(t) = t^{\frac{1}{4}} e^{\frac{2}{3}t^{3/2}} \quad ; \quad (4,09)$$

$$v(t) = \frac{1}{2} t^{-1/4} e^{-\frac{2}{3}t^{2/3}} \quad ; \quad v'(t) = -\frac{1}{2} t^{\frac{1}{4}} e^{-\frac{2}{3}t^{3/2}}. \quad (4,10)$$

From the series $(4,04)$ it is not difficult to extract the formula

$$w\left(t e^{i\frac{\pi}{3}}\right) = 2e^{i\frac{\pi}{6}} v(-t), \qquad (4,11)$$

$$w\left(t e^{i\frac{2\pi}{3}}\right) = e^{i\frac{\pi}{3}} [u(t) - iv(t)], \qquad (4,12)$$

which give an impression of the behaviour of $w(t)$ in a complex plane.

We note that the function $w(t)$ is expressed via the Hankel function of the order $1/3$ by the formula

$$w(t) = \sqrt{\frac{\pi}{3}} e^{i\frac{2\pi}{3}} (-t)^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3}(-t)^{3/2}\right) \qquad (4,13)$$

Having learned the major properties of $w(t)$, we switch to the derivation of the asymptotic expression for the Hankel function $H_{\nu}^{(1)}(\rho)$, where ν and ρ are large and close to each other, so that the relationship

$$\frac{\nu - \rho}{\sqrt{\rho/2}} = t \qquad (4,14)$$

remains limited.

The Hankel function $H_\nu^{(1)}(\rho)$ assumes the integral representation

$$H_\nu^{(1)}(\rho) = \frac{1}{\pi i} \int_C e^{-\rho \, \text{sh} \, v + \nu v} \, dv, \qquad (4,15)$$

where the contour C goes along the straight line $\text{Im}(v) = -\pi$ from $-\pi i - \infty$ to a certain point $v = v_0$ (e.g. $v_0 = -\pi/\sqrt{3} - i\pi$), then along the straight line from $v = v_0$ to $v = 0$ and, finally, along the material axis from zero to infinity. We express, in accordance with (4,14), ν through t and introduce the variable integration

$$z = \sqrt[3]{\frac{\rho}{2}} \, v. \qquad (4,16)$$

Considering t and z to be finite and ρ to be large, we may displace the subintegral function in (4,15) according to the negative (fractional) power of ρ. Since in the main part the transformed S contour coincides with the Γ contour, we can write

$$H_\nu^{(1)}(\rho) = \frac{1}{\pi i} \left(\frac{\rho}{2} \right)^{-\frac{1}{3}} \int_\Gamma e^{tz - \frac{1}{3}z^3} \left[1 - \frac{1}{60} \left(\frac{\rho}{2} \right)^{-\frac{2}{3}} z^5 + \ldots \right] dz, \qquad (4,17)$$

and calculating the integrals with the aid of (4,01)

$$H_\nu^{(1)}(\rho) = -\frac{i}{\sqrt{\pi}} \left(\frac{\rho}{2} \right)^{-1/3} \left\{ w(t) - \frac{1}{60} \left(\frac{\rho}{2} \right)^{2/3} w^{(5)}(t) + \ldots \right\}. \qquad (4,18)$$

By virtue of the differential equation (4,02), the 5th derivative equals

$$w^{(5)}(t) = t^2 w'(t) + 4 t w(t). \qquad (4,19)$$

ubstituting this expression in (4,18) and going over by Formula
(.,11) to the function $\zeta_{v-\frac{1}{2}}(\rho)$, we shall have

$$\zeta_{v-\frac{1}{2}}(\rho) = - i \left(\frac{\rho}{2}\right)^{1/6} \left\{ w(t) - \frac{1}{60} \left(\frac{\rho}{2\cdot}\right)^{-2/3} [(t\ w'(t) + 4tw(t)] + \ldots \right\}.$$

$$(4,20)$$

Differentiating this expression according to ρ when v is constant
and taking into account the dependence of t on ρ, we get the following
formula for the derivative:

$$\zeta'_{v-\frac{1}{2}}(\rho) =$$

$$= i \left(\frac{\rho}{2}\right)^{-1/6} \left\{ w'(t) - \frac{1}{60} \left(\frac{\rho}{2}\right)^{-2/3} [(t^3 + 9)w(t) - 4tw'(t)] + \ldots \right\}.$$

$$(4,21)$$

We shall be making use of these expressions later.

5. EXPRESSION OF THE HERTZ FUNCTION APPLICABLE IN THE TWILIGHT REGION

We transcribe Expression (3,02) for the Hertz function, replacing in it
the value G_v^* by the approximate value $\sqrt{\pi/v}$ and the value $\sin \vartheta$ in front of
the integral by the approximate value ϑ. We obtain

$$U_a = \frac{2e^{i\frac{3\pi}{4}}}{kab\ \sqrt{2\pi\vartheta}} \int_C \varphi(v)e^{iv\vartheta} \sqrt{v}dv .$$

$$(5,01)$$

By the C contour we shall understand the C_2 contour described in Para.2,
or some equivalent of it.— The main part of the integration will, in our case
(i.e. for finite values of the p parameter), lie close to the point where
$v = ka$.

The function $x_{v-\frac{1}{2}}(k,a)$ appearing in the expression (2,02) for $\varphi(v)$, we
can now replace, therefore, by the value of the expression (3,08) when $v = ka$,
after which we shall have

$$\varphi(v) = - \frac{\zeta_{v-\frac{1}{2}}(kb)}{\zeta^1_{v-\frac{1}{2}}(ka) + i\frac{k}{k_2} \sqrt{1 - \frac{k_2}{k_2^2}} \zeta_{v-\frac{1}{2}}(ka)} .$$

$$(5,02)$$

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For the function $\zeta_{\nu-\frac{1}{2}}$ and its derivative we must take expressions which will be correct in the proximity of $\nu = ka$. Such expressions were obtained in the preceding paragraph. Leaving the main terms in (4,20) and (4,21), we shall have

$$\zeta_{\nu-\frac{1}{2}}(ka) = -i\left(\frac{ka}{2}\right)^{1/6} w(t). \qquad (5,03)$$

$$\zeta'_{\nu-\frac{1}{2}}(ka) = i\left(\frac{ka}{2}\right)^{-1/6} w'(t), \qquad (5,04)$$

where t is linked to ν by the relationship

$$\nu = ka + \left(\frac{ka}{2}\right)^{1/3} t. \qquad (5,05)$$

The numerator in (5,02) is obtained from (5,03) by substituting a for b and t for t', where

$$\nu = kb + \left(\frac{kb}{2}\right)^{1/3} t'. \qquad (5,06)$$

Comparing (5,05) and (5,06) we get a link between t and t'. But the relationship h/a, where $h = b - a$, is a small value (we will consider it to be of the same order as $(ka)^{-2/3}$). Ignoring this value in comparison with unity, we are able to write

$$t' = t - y, \qquad (5,07)$$

where

$$y = \frac{kh}{\left(\frac{ka}{2}\right)^{1/3}} \qquad (5,08)$$

is a value proportional to the height of the source above the Earth. The value y may be called the accepted altitude of the source. Thus, with accuracy to terms of the order of h/a or $(ka)^{-2/3}$, will be

$$\zeta_{\nu-\frac{1}{2}}(kb) = -i \left(\frac{ka}{2} \right)^{1/6} w(t - y),$$ (5,09)

where t is determined from (5,05) (in the multiple before $w(t-y)$ we similarly substitute b for a).

Substitutions (5,03), (5,04) and (5,09) in (5,02) give the approximate expression for $\varphi(\nu)$.

If we assume for brevity

$$q = i \left(\frac{ka}{2} \right)^{1/3} \frac{k}{k_?} \sqrt{1 - \frac{k^2}{k_?^2}},$$ (5,10)

we shall have

$$\varphi(\nu) = - \left(\frac{ka}{2} \right)^{1/3} \frac{w(t - y)}{w'(t) 1 qw(t)}.$$ (5,11)

Recalling the formulae (1,09) and (1,10) we can write the value q in the form

$$q = i \left(\frac{\pi a}{\lambda} \right)^{1/3} \frac{\sqrt{\varepsilon - 1 + i\dfrac{\lambda}{2\pi l}}}{\varepsilon + i \dfrac{\lambda}{2\pi l}}$$ (5,12)

or, with the same accuracy

$$q = i \left(\frac{\pi a}{\lambda} \right)^{1/3} \frac{1}{\sqrt{\varepsilon + 1 + i\dfrac{\lambda}{2\pi l}}}.$$ (5.13)

The last expression is rather more convenient for calculation.

It remains for us now to substitute the value $\varphi(v)$ from (5,11) in the formula (5,01) and transfer it to the variable integration t. Making this substitution, we can exchange the value $\sqrt{v}$ under the integral with the constant value $\sqrt{ka}$, and also write a in place of b in the multiple before the integral. As a result we obtain a formula which may be written in the form

$$U_a = \frac{e^{ika\vartheta}}{a\vartheta} e^{-i\frac{\pi}{4}} \sqrt{\frac{\chi}{\pi}} \int_C e^{i\chi t} \frac{w(t-y)}{w'(t) - qw(t)} \, dt, \qquad (5,14)$$

where letter χ signifies, for brevity, the amplitude

$$\chi = \left(\frac{ka}{2} \right)^{1/3} \vartheta , \qquad (5,15)$$

which can be called the applied horizontal distance from the source, such that y and q have the value (5,08) and (5,13). Countour C must embrace all the poles of the subintegral function, as we shall see, all of them are located in the first quarter of plane t.

In order graphically to present the relationship of the horizontal and vertical scales in the variables χ and y, we shall compose an expression for the parameter p, determined by the formula (3,09). From examination of a triangle with its angles at the centre of the Earth, the source and the point of observation, it is not difficult to extract the approximate formula

$$p = \left(\frac{ka}{2} \right)^{\frac{1}{3}} \cos \gamma = \frac{y - \chi^2}{2\chi} . \qquad (5,16)$$

Thus, the equation for the horizon line is $\chi = \sqrt{y}$. Later we shall require a link between the distance R from the source, considered along a straight line, and the horizontal distance $a\vartheta \gg h$, i.e. $(ka)^{\beta} \chi \gg y$, this link has the form:

$$kR = ka\vartheta + \chi_o, \qquad (5,17)$$

where

$$\omega_0 = \frac{y^2}{4\chi} + \frac{\chi y}{2} - \frac{\chi^3}{12} .$$ (5,18)

6. STUDY OF THE EXPRESSION FOR THE HERTZ FUNCTION

The expression (5,14) found for the Hertz function, may conveniently be written in the form

$$U_a = \frac{e^{ika\vartheta}}{a\vartheta} \cdot V(\chi, y, q),$$ (6,01)

where

$$V(\chi, y, q) = e^{-i\frac{\eta}{4}} \sqrt{\frac{\chi}{\pi}} \int_C e^{i\chi t} \frac{w(t - y)}{w'(t) - qw(t)} \, dt.$$ (6,02)

The value V, by analogy with the value W introduced earlier in (3,14), can be called the attenuation factor. We shall establish the connection between V and W. Since, in the denominators of Formulae (3,13) and (6,01) we can consider the values of R and $a\vartheta$ as being equal one to another, we sha'l obtain, in consequence of (5,17):

$$W = Ve^{-i\omega_0}.$$ (6,03)

We now have to examine the expression (6,02) for V. We shall begin this examination at the case where the value p is positive and large (illuminated region). We have already examined this case from a different approach (Para. 3). However, since formula (6,02) was brought out by us for finite p, it is an interesting conclusion that it is true also for large values of p.

When p $\sim$ 1, the main part of the integration will lie with the larger negative t (namely close to $\overline{t} = p$). Using Expressions (4,07) and (4,08) for w and w' and using the fixed phase method, we obtain

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$$\eta = e^{i\omega_o} \frac{2}{1 - i\frac{q}{p}} \tag{6.04}$$

and, as a result of (6,03),

$$W = \frac{2}{1 - i\frac{q}{p}} \tag{6,05}$$

The last expression practically coincides with (3,15). We note that if x is of the order of unity or is large, then for the applicability of the fixed phase method the condition $p \gg 1$ suffices; if x is small then it is necessary that $y' \gg 2x$. If the latter condition is not fulfilled, but the condition

$$x < y \ll \frac{1}{x}; \tag{6,06}$$

is, then the integral (6,02) may be calculated by a different method. In the asymptotic expression for $w(t - y)$ a further simplification may be made, after which the integral (6,02) takes the form

$$V = e^{i\frac{\pi}{4}} \sqrt{\frac{}{}} \int_{C} \frac{e^{i \cdot t + iy \cdot \overline{-t}}}{\sqrt{-t - iq}} \, dt. \tag{6,07}$$

Having taken as a variable integration the value $\sqrt{-t}$, we arrived at an integral in the form (3,20) in which $\left[\sqrt{-t} = \left(\frac{k\alpha}{2}\right)_\mu^{1/3}\right]$ and once again get the Weyl - van der Pol formula with values of σ and τ, equal to

$$e^{-i\frac{\pi}{4}} q \qquad \qquad = e^{i\frac{\pi}{4}} \cdot \frac{y}{2\sqrt{x}} \tag{6,08}$$

and practically coincide with (3,22).

We now proceed to a more interesting condition, when the value of ρ is a positive or negative number of the order of unity. As we know, this is the twilight region, where the diffraction phenomena plays a major role.

If the value of χ and y are of the order of unity, then the most convenient method of computing the integral (6,02) is to present it in the form of the sum of deductions relating to the poles of the subintegral function.

If $t_s = t_s(q)$ signifies the roots of the equation

$$w'(t) - qw(t) = 0 \qquad\qquad (6,09)$$

then we shall have

$$V(\chi, y, q) = e^{i\frac{\pi}{4}} 2\sqrt{\pi\chi} \sum_{s=1}^{\infty} \frac{e^{i\chi t_s}}{t_s - q^2} \frac{wt_s - y}{w(t_s)} , \qquad (6,10)$$

The roots $t_s(q)$ are the essence of the function from the complex parameter p. When $q = 0$ they revert to the root $t_s' = t_s(0)$ of the derivative $w'(t)$, but when $q = \infty$ they revert to Root $t_s^O = t_s(\infty)$. The values t_s' and t_s^O have Phase $\pi/3$, so that

$$t_s' = | t_s' | e^{i\frac{\pi}{3}} ; \qquad t_s^O = | t_s^O | e^{i\frac{\pi}{3}} \qquad (6,11)$$

We show here a module of the first few roots

s	$|t_s|$	$|t_s^O|$
1	1.01879	2.33811
2	3.24820	4.08795
3	4.82010	5.52056
4	6.16331	6.78671
5	7.37218	7.94417

At large s values will be

$$|t_s'| \approx \left[\frac{3\pi}{2}\left(s - \frac{3}{4}\right)\right]^{2/3}; \quad |t_s^0| \approx \left[\frac{3\pi}{2}\left(s - \frac{1}{4}\right)\right]^{2/3} \quad (6,12)$$

At finite values of q, one may compute the roots using the differential equation

$$\frac{dt_s}{dq} = \frac{1}{t_s - q^2}, \quad (6,13)$$

which is easily extracted from (4,03). Root t can be determined either as the solution to (6,13), which, when $q = 0$, reverts to t_s', or as the solution which, when $q = \infty$, reverts to t_s^0; both solutions coincide. From the first definitio it is easy to construct for t_s a series according to rising degrees of q; it will converge when $|q| < \sqrt{t_s}|$. From the second definition it is possible to construct a series according to falling (negative) degrees of q; it will converge where $|q| > |\sqrt{t_s}|$. We are not reproducing these series here. We note that the value t, which at large values of $|q|$ is close to q^2, is not a root of Equation (6,09).

In conditions where $y^2 \ll 2 |\sqrt{t_s}|$ we have the approximate equation

$$\frac{w(t_s - y)}{w(t_s)} = \mathrm{ch}\,(y\sqrt{t_s}) - \frac{q}{\sqrt{t_s}}\,\mathrm{sh}\,(y\sqrt{t_s}), \quad (6,14)$$

which permits evaluation of the remote terms of series (6,10). At large s values, such that $|q| \ll |\sqrt{t_s}|$, approximately t_s will equal $t_s(0)$ which will equal t_s'. From here and from (6,14) it follows that the series (6,10) always converges. However, if χ is small or y large, then to compute the sum of the series may require a large number of terms.

In the shaded region, when the value of p is large and negative, the series (6,10) converges very quickly and its sum approximately boils down to the first term.

Our series (6,10) corresponds to the Watson series, but has the advantage that the terms of our series have simple expressions.

Our basic formula (6,02) permits investigation not only of extreme conditions (large positive p - the illuminated region, large negative p - the shaded region) but also the intermediate conditions, in particular the twilight region. While for the extreme conditions our formula does no more than clarify earlier known formulae (reflecting formula and Weyl - van der Pol formula for the illuminated region and the Watson series for the shaded region), for the twilight region it gives essentially new results.

Particular interest is presented by the case where χ and y are large, while p is finite (short waves, twilight). This condition has not been investigated by anyone to date, and previously known formulae are inapplicable to it. We shall derive here the approximate formulae which facilitate its study.

We introduce the value

$$z = \chi - \sqrt{y},\qquad (6,15)$$

which is the reduced distance, calculated not from the source but from the geometric shadow boundary. In the region of geometric shadow $z > 0$, in the "visible" region $z < 0$. We have

$$p = \frac{y - \chi^2}{2\chi} = -z + \frac{z^2}{2\chi}.\qquad (6,16)$$

In our assumptions χ is large, while z is finite; therefore we shall have approximately $p = -z$.

In the case under examination the main part of the integration in Integral (6,02) will correspond to values of t of the order of unity. But with y large and t finite, the asymptotic expression (4,07) will apply to the function $w(t - y)$, which will give

$$w(t - y) = e^{i\frac{\pi}{4}}(y - t)^{-\frac{1}{4}} e^{i\frac{2}{3}(y-t)^{3/2}}\qquad (6,17)$$

or approximately

$$w(t - y) = e^{i\frac{\pi}{4}} y^{-\frac{1}{4}} e^{i\frac{2}{3}y^{3/2} - i\sqrt{y}\,t}.\qquad (6,18)$$

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Substituting (6,18) in (6,02) and exchanging the value $x^{\frac{1}{2}}y^{\frac{1}{4}}$ for unity in the multiple in front of the integral, we shall have

$$V(x,y,q) = e^{i\frac{2}{3}y^{3/2}} V_1(x - \sqrt{y}, q).$$

(6,19)

where

$$V_1(z,q) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{izt}}{w'(t) - qw(t)} dt.$$

(6,20)

The terms discarded in (6,19) will, with finite z values, be of the order of $1/\sqrt{y}$ (or $1/x$).

Thus, in our case, the function $V(x,y,q)$ from two arguments and from Parameter q will come down to the function $V(z,q)$ from one argument and from the very same parameter. This is a considerable simplification.

We shall define the formula linking the attenuation function W to the value V. We have the identity

$$\frac{2}{3} y^{3/2} = \omega_0 + \frac{1}{3} z^3 - \frac{z^4}{4x}$$

(6,21)

where ω_0 has the value (5,18). Discarding in (6,21) the last term, we get from (6,03) and (6,19)

$$W = e^{\frac{i}{3}z^3} V_1(z,q)$$

(6,22)

In this way, in our approximation, the attenuation function W depends on x and y only owing to $z = z - \sqrt{y}$.

The function $V_1(z,q)$ is the whole transcendental function from the variable z. For positive z values the integral (6,20) can be computed as the sum of deductions, which gives

$$V_1(z,q) = i\,2\,\sqrt{\pi}\;\sum_{s=1}^{\infty}\frac{e^{\,izt_s}}{(t_s - q\,)w(t_s)} \qquad \text{(where } z > 0, \quad (6,23)$$

where t_s - are the previously investigated roots of the equation (6,09).
Series (6,23) converges more quickly the larger z is. At large positive values
of z its sum boils down to the first term. For finite negative z (e.g. for
- 2 < z < 0) Integral (6,20) must be computed by squaring. For large negative
values of z one may compute the integral by the fixed phase method, at which
one obtains

$$V_1(z, q) = \frac{2e^{-\frac{i}{3}z^3}}{1 + \frac{iq}{z}} \qquad (6,24)$$

and, as a result of (6,22)

$$W = \frac{2}{1 + \frac{iq}{z}} . \qquad (6.25)$$

Keeping in mind that the approximation z = - p, we get a coincidence with
Formula (6,03).

In conclusion we note that our basic formula (6,02) may also be obtained
by a method of parabolic relation by M.A. Leontovich and applied by him[5] to
the derivation of the Weyl - van der Pol formula. Use of Leontovich's method
(with certain refinements), applied to this problem, will be examined by us in
a special article.

SUMMARY

The problem of the propagation of radio waves around the homogeneous
surface of the earth is investigated. The diffraction effects are considered
but the influence of the ionosphere is neglected. The aim of the paper is
to derive formulae for the wave amplitude as a function of the elevation of
the source, its distance from the point of observation (situated on the surface
of the earth), of the wave length and of the electrical properties of the soil.

The main result is the derivation of an expression for the attenuation factor of an integral. This expression is valid for all the values of parameters, which are of practical interest. In the limiting cases the well known formulae are obtained: The Weyl - van der Pol formula for the illuminated region and the formula which corresponds to the first term in Watson's series for the shaded region (the latter in a slightly corrected form). Essentially new is the investigation of the region of the continuous transition from the illuminated region to the shaded one. Methods for numerical calculations of sums and integrals involved in the problem are elaborated.

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